HON 329                                                                        Fall 2011
Paradoxes and the Creation of Set Theory in the 20th Century

Catalog Description:
Prerequisite, Math 104, and acceptance to the University Honors Program, or consent of instructor. Set theory has been described as one of the most beautiful areas of mathematics, but it was born out of the necessity of answering some puzzling paradoxes dealing with infinity. In fact, the theory itself is fraught with paradoxical statement. Cantor himself, when he discovered that the real line and the real plane had the same cardinality, wrote to Dedekind: “I see it, but I don’t believe it.” By the same token, it was Hausdorff who, in a later work, wrote that set theory is a field “in which nothing is by itself evident, and true statements are often paradoxical, and plausible statements are false.” This course will explore some of these paradoxes at the origins of set theory and will discuss in detail the notions of infinity, as well as of cardinal and ordinal numbers. The instructor will use both intuitive arguments, as well as rigorous proofs, in an attempt to both convince the student, and rigorously justify the statements presented. (Offered as needed.) 3 credits.

Course Objectives:
1. Develop an understanding for the intellectual significance of set theory, mathematical logic, and of mathematical ideas in general.
2. Develop the skill to prove fundamental properties using instruments such as mathematical induction, and set theoretical arguments.
3. Learn the foundations of modern set theory, including the Axiom of Choice, the notion of cardinal numbers, the notion of ordinal numbers.

Content:
Students will learn the fundamental ideas of modern set theory, and will be able to demonstrate rigorously many of the fundamental properties of infinite sets. After reviewing basic notions of set theory (elementary operations on sets, functions and relations, equivalence and order relations), we will define finite sets and we will offer two different definitions of infinity. We will show that, with the use of the Axiom of Choice, these two definitions are equivalent. We will then define the notion of equipotent sets and the notion of cardinal numbers. As we do so, we will deal with
the classical Russel paradox, and we will illustrate several ways to avoid it. As an application of the notion of cardinal numbers, we will study countable sets, and we will show that the set of real numbers is uncountable. We will also see that the set of real numbers is equipotent to the set of real points in the plane. We will formulate the Continuum Hypothesis, both in its traditional form, and in its generalized one. We will introduce the arithmetic of cardinals (addition and multiplication, noticing again the role played by the Axiom of Choice in order for these definitions to be well behaved), and we will introduce an order among cardinals: this part of the course will culminate with the proof of the Equivalence Theorem, that establishes that the ordering we have introduced satisfies the properties of, at least, a partial order. For us to be able to prove that it is in fact a total order, we will then introduce the notion of transfinite type, of well-ordering, and of ordinal number. We will study the arithmetic of such numbers, and the fact that a total order can be given to the set of ordinal numbers. We will show that the induction property which is so important in the theory of natural numbers, can be extended to ordinal numbers by what is called transfinite induction. Finally, we will go back to the Axiom of Choice, we will use it to prove the Well-Ordering Theorem, which will allow us to show that the order we originally introduced among cardinal numbers is a total order. This will bring us to close the circle and will conclude the class.

**Current Required Texts:**
Instructor will distribute materials

**Instructional Strategies:**
Seminar style class. The course is proof intensive, and students will be asked to prove statements of increasing difficulty throughout the course, as part of their homework assignments.

**Methods of Evaluation:**
Mostly class interaction, homework, and a midterm test.

**Chapman University Academic Integrity Policy:**
The course syllabus should include the following statement:
Chapman University is a community of scholars which emphasizes the mutual responsibility of all members to seek knowledge honestly and in good faith. Students are responsible for doing their own work, and academic dishonesty of any kind will not be tolerated anywhere in the university

**Students with Disabilities Policy:**
The course syllabus should include the following statement:
In compliance with ADA guidelines, students who have any condition, either permanent or temporary, that might affect their ability to perform in this class are encouraged to inform the instructor at the beginning of the term. The University, through the Center for Academic Success, will work with the appropriate faculty member who is asked to provide the accommodations for a student in determining what accommodations are
suitable based on the documentation and the individual student needs. The granting of any accommodation will not be retroactive and cannot jeopardize the academic standards or integrity of the course.

**Prepared by:**
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**Last revised:**
Daniele Struppa, November 2011