The anabelian geometry of Grothendieck

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Grothendieck 1928-2014

- Mathematician ahead of his time
- Philosopher mathematician
- Isolated mathematician
- Outstanding contribution to Galois theory

Mathematical questions

- Analysis/Topology versus Algebra ?
- What is the precise gap between commutative and non-commutative mathematics ?

Évariste Galois 1811-1832

$f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0 \in \mathbb{Q}[X]$

$\mathrm{Gal}(f)$ finite group

$f(X)$ solvable by radicals $\iff$ $\mathrm{Gal}(f)$ solvable

In general K field $\overline{K} = K^{\mathrm{alg}}$

$$G_K = \mathrm{Gal}(\overline{K}/K) \stackrel{\mathrm{def}}{=} \mathrm{Aut}(\overline{K}/K)$$

Functor $\{\text{Fields}\} \xrightarrow{\mathrm{Gal}} \{\text{Profinite Groups}\}$

- What is the image of Gal ?

Fact $G_{\mathbb{Q}}$ **unknown**

G_K K infinite fin. gen. **mysterious !**

Various approaches $K/\mathbb{Q}$ finite

- Class Field Theory: explicit description of G_K^{ab}
- Iwasawa: understand G_K^{metab}
- Inverse Galois Problem: Hilbert, Shafarevich, ...
- Galois representations: Weil, Shimura, Serre, Deligne, Faltings, Wiles, $\cdots$
- Langlands programme: L-functions, automorphic forms, representation

Grothendieck SGA1 1960

X connected scheme $\rightsquigarrow$ $\pi_1(X,*)$

$X \to \operatorname{Spec} K$ algebraic variety

(*) $$1 \to \pi_1(X)^{\text{geo}} \to \pi_1(X) \to G_K \to 1$$

(**) $$\rho_X : G_K \to \text{Out}\ (\pi_1(X)^{\text{geo}})$$

X proper smooth **curve** genus$(X) = g$

$$\Gamma_g = \frac{<a_i, b_i>_{i=1}^g}{\prod_{i=1}^g [a_i, b_i]} \qquad \Gamma_g^\wedge$$

- $\text{char}(K) = 0$ $\qquad \pi_1(X)^{\text{geo}} \overset{\sim}{\to} \Gamma_g^\wedge$
- $\text{char}(K) = p > 0$ (Weil)

$$\pi_1(X)^{\text{geo},(p')} \overset{\sim}{\to} \Gamma_g^{\wedge,(p')}$$

Fundamental examples $K = \mathbb{Q}$

- $X = E$ elliptic curve $\pi_1(X)^{\text{geo}} \overset{\sim}{\to} \hat{\mathbb{Z}}^2$

$$\rho_X : G_{\mathbb{Q}} \to GL_2(\hat{\mathbb{Z}})$$

- $X = E \setminus \{0\}$ $\pi_1(X)^{\text{geo}} \overset{\sim}{\to} F_2$

$$\rho_X : G_{\mathbb{Q}} \to \text{Out}(F_2)$$

- $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ $\pi_1(X)^{\text{geo}} \overset{\sim}{\to} F_2$

$$\rho_X : G_{\mathbb{Q}} \to \text{Out}(F_2)$$

Grothendieck 1966: proof of Fermat?

Grothendieck anabelian conjectures (1980's)

K fin. gen. $\mathrm{char}(K) = 0$

- **AN1** L, F fin. gen. over K

$$\mathrm{Hom}_K(F, L) \to \mathrm{Hom}_{G_K}(G_L, G_F)/\sim$$

is a bijection

- **AN2** X, Y hyperbolic K-curves

$$\mathrm{Hom}_K(X, Y) \to \mathrm{Hom}(\pi_1(X), \pi_1(Y))/\sim$$

is a bijection

- **Tate conjecture**: A, B abelian varieties over K

$$\mathrm{Hom}_K(A, B) \otimes \hat{\mathbb{Z}} \to \mathrm{Hom}_{G_K}(\pi_1(A)^{\mathrm{geo}}, \pi_1(B)^{\mathrm{geo}})$$

is a bijection

- Arithmetic + topology $\Longrightarrow$ rigid situation !

- **AN1** (isom form): Neukirch-Uchida (1970)

Pop, Spiess (1990's)

- **AN2** (isom form): Nakamura, Tamagawa,

Mochizuki (1990's)

- Mochizuki (1990's): **AN1, AN2**, K sub-p-adic field (p-adic Hodge theory)

$$\{\text{Fin. Gen. Fields }\} \xrightarrow{\text{Gal}} \{\text{Profinite Groups}\}$$

$$\{\text{Hyp. Curves }\} \xrightarrow{\pi_1} \{\text{Profinite Groups}\}$$

- Images of "Gal" and "π_1" functors **mysterious**

Aim Improve this situation !

What is the meaning of **anabelian**?

Nakamura, Tamagawa, Mochizuki:

Here, the term **"anabelian algebraic variety"** means roughly "an algebraic variety whose geometry is controlled by its fundamental group, which is assumed to be **'far from abelian'** ".

False intuition !

What is the **right anabelian geometry** ?

May 2017 (*)

m-step solvable anabelian geometry

G profinite group

- $\ldots \subseteq G(i+1) \subseteq G(i) \subseteq \ldots \subseteq G(1) \subseteq G(0) = G$

  $G(i+1) = \overline{[G(i), G(i)]} \qquad i \geq 0$

- $G^i \stackrel{\text{def}}{=} G/G(i) \qquad$ i-th **step solvable quotient** of G

$G^1 = G^{\text{ab}}$, $G^2 = G^{\text{metab}}, \cdots$

$j > i$:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G(i) & \longrightarrow & G & \longrightarrow & G^i & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \text{id} & & \\
1 & \longrightarrow & G[j,i] & \longrightarrow & G^j & \longrightarrow & G^i & \longrightarrow & 1
\end{array}$$

- $K/\mathbb{Q}$ finite $\qquad m \geq 1$

Fact Structure of G_K^m can be approached via CFT (in principle)

(Saïdi-Tamagawa 2017-2019)

Theorem A: L, F fin. gen. $\dim(F) = \dim(L) = d$

$$\mathrm{Isom}(F, L) \to \mathrm{Isom}(G_L^m, G_F^m)/\sim$$

is a bijection for all $m \geq d^2 + 4d - 2$.

Expected B: X, Y hyperbolic curves (fin. gen. fields)

$$\mathrm{Isom}(X, Y) \to \mathrm{Isom}(\pi_1^m(X), \pi_1^m(Y))/\sim$$

is a bijection for all $m \geq 3$.

$$\pi_1(X)^m \twoheadrightarrow G_K^m$$

Facts

- No need to know G_K in anabelian geometry !
- Theorem A **reconciles** anabelian geometry with CFT
- **Anabelian** world close to **abelian** world !

Mathematical philosophy of Grothendieck

What my experience of mathematical work has taught me again and again, is that the **proof always springs from the insight**, and not the other way round ? and that the insight itself has its source, first and foremost, in a **delicate and obstinate feeling of the relevant entities and concepts and their mutual relations**. The guiding thread is the inner coherence of **the image which gradually emerges from the mist**, as well as its consonance with what is known or foreshadowed from **other sources** - and it guides all the more surely as the "exigence" of coherence is stronger and more delicate.

How can we benefit more from Grothendieck Today ?

- Embrace more his mathematical philosophy in our way of doing research in mathematics, and reconcile his philosophy with "practical mathematics".

- Embrace more his mathematical philosophy in our way of teaching mathematics.