Vector bundles on Compact Riemann surfaces according to Grothendieck and his followers

Oumar Wone

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Introduction

Our goal in this presentation is firstly to present the theorem of Birkhoff-Grothendieck[a] which classifies vector bundles on the Riemann sphere. Then we will talk about further works concerning the classification of vector bundles on compact Riemann surfaces/complex manifolds, not necessarily in a chronological order, but maybe in a logical order. For instance we will touch on the works of Atiyah[b],[c],[d].

[a] A. Grothendieck. "Sur la classification des fibrés holomorphes sur la sphère de Riemann". In: *Am. J. Math.* 79 (1957), pp. 121–138.

[b] M. F. Atiyah. "Complex analytic connections in fibre bundles". In: *Trans. Am. Math. Soc.* 85 (1957), pp. 181–207.

[c] M. F. Atiyah. "Vector bundles over an elliptic curve". In: *Proc. Lond. Math. Soc.* (3) 7 (1957), pp. 414–452.

[d] M. F. Atiyah. "On the Krull-Schmidt theorem with application to sheaves". In: *Bull. Soc. Math. Fra.* 84 (1956), pp. 307–317.

Definition

A Riemann surface M is a connected one dimensional complex manifold, i.e. a two dimensional real smooth manifold with a maximal set of coordinate charts $\phi_\alpha : U_\alpha \to \mathbb{R}^2 \simeq \mathbb{C}$: such that the change of charts $\phi_\beta \circ \phi_\alpha^{-1}$, is an invertible holomorphic function from $\phi_\alpha(U_\alpha \cap U_\beta)$ to $\phi_\beta(U_\alpha \cap U_\beta)$, for all α, β, and M is connected.

Example

Recall that $\mathbb{P}^1(\mathbb{C})$ is the set lines of $\mathbb{C}^2$ through the origin: For $(z_0, z_1) \in \mathbb{C}^2 \setminus \{(0,0)\}$ we define

$$[z_0 : z_1] = \{\lambda(z_0, z_1), \lambda \in \mathbb{C}\}$$

then

$$\mathbb{P}^1(\mathbb{C}) := \{[z_0 : z_1], (z_0, z_1) \in \mathbb{C}^2 \setminus \{(0,0)\}\}.$$

Let $M = \mathbb{P}^1(\mathbb{C})$ be the usual coordinate patches $U_0 = \{[z_0 : z_1] \in \mathbb{P}^1(\mathbb{C}),\ z_0 \neq 0\}$ and $U_1 = \{[z_0 : z_1] \in \mathbb{P}^1(\mathbb{C}),\ z_1 \neq 0\}$.

Example (continued)

Then we have the following two homeomorphisms

$$\psi_0 : U_0 \to \mathbb{C},\ [z_0 : z_1] \mapsto \frac{z_1}{z_0}$$

and

$$\psi_1 : U_1 \to \mathbb{C},\ [z_0 : z_1] \mapsto \frac{z_0}{z_1};$$

on the overlap we have

$$\psi_1 \circ \psi_0^{-1} : \mathbb{C}^\times \to \mathbb{C}^\times,\ \ z \mapsto \frac{1}{z},$$

which is holomorphic.

Definition

A complex manifold M of dimension $m \geqslant 1$, is defined similarly to a Riemann surface, but this time the transition functions are invertible holomorphic maps from an open set of $\mathbb{C}^m$ to an open set of $\mathbb{C}^m$.

Definition

Let M be a complex manifold of dimension $m \geqslant 1$ and N be a complex manifold of dimension $n \geqslant 1$. A holomorphic map $f : N \to M$ is a continuous map such that for each coordinate chart $\phi_\alpha : U_\alpha \to \mathbb{C}^m$ on M and each chart $\psi_\beta : V_\beta \to \mathbb{C}^n$ on N, such that $f(V_\beta) \subset U_\alpha$ we have that

$$\phi_\alpha \circ f \circ \psi_\beta^{-1}$$

is holomorphic.

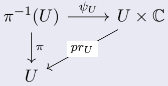
Definition

A holomorphic line bundle L over a Riemann surface M is a two dimensional complex manifold L with a holomorphic projection

$$\pi : L \to M$$

such that

- for each $m \in M$, $\pi^{-1}(m)$ has the structure of a one-dimensional vector space,
- each point $m \in M$ has a neighborhood U and a homeomorphism ψ_U such that



Definition (continued)

is commutative

- $\psi_V \circ \psi_U^{-1}$ is of the form

$$(m, w) \mapsto (m, g_{VU}(m)w)$$

where g_{VU} is a non-vanishing holomorphic function. The functions g_{VU} are called the transition functions of the line bundle and ψ_U is a local trivialization over U of the line bundle.

Example

Take $p \in M$, U_0 a neighborhood of p with coordinate chart z such that $z(p) = 0$. Let $U_1 = M \setminus \{p\}$. Then we can use z as a transition function to define a line bundle on M since $z = g_{01}$ is holomorphic and non-vanishing on $U_0 \cap U_1 = U_0 \setminus \{p\}$. We patch together $U_0 \times \mathbb{C}$ and $U_1 \times \mathbb{C}$ over $U_0 \cap U_1$ by using ψ defined by

$$\psi(m, w) = (m, g_{01}(m)w).$$

This gives for each point $p \in M$ a line bundle which we denote by L_p, called the point bundle associated to p.

Definition

A holomorphic section of a holomorphic line bundle L over M is a holomorphic map $s : M \to L$ such that $\pi \circ s = \mathrm{Id}_M$.

In local trivializations ψ_U, ψ_V of the line bundle L, the section gives

$$\psi_U(s(m)) = (m, s_U(m)) \in U \times \mathbb{C}$$

and

$$\psi_V(s(m)) = (m, s_V(m)) \in V \times \mathbb{C}$$

hence on the overlap $U \cap V$, we have

$$\psi_V \circ \psi_U^{-1}(m, s_U(m)) = (m, s_V(m)) = (m, g_{VU} s_U(m)).$$

This gives

$$s_V = g_{VU} s_U, \text{ on } U \cap V.$$

One can add sections pointwise

$$(s+t)(m) := s(m) + t(m)$$

and multiply sections by scalars

$$(\lambda s)(m) = \lambda s(m),$$

so the space of all sections of L is a vector space, denoted $H^0(M, L)$.

Theorem

If M is a compact Riemann surface, $H^0(M, L)$ is finite dimensional.

Example

- The line bundle L_p has a canonical section s_p: take the two functions z on $U_0 \ni p$ and 1 on $U_1 = M \setminus \{p\}$, then we get a section of L_p since

$$z = z \times 1 = g_{01} 1.$$

  The section s_p has a simple zero at p and only there.
- The canonical bundle K: bundle of holomorphic 1-forms. Suppose we have local coordinates z, w, with $w(z) = \phi_\beta \circ \phi_\alpha^{-1}(z)$, a function of z on the overlap. Then dz and dw give local trivializations of the canonical bundle, and on the overlap $dw = \frac{dw}{dz} dz$. This gives the transition functions $\frac{dw}{dz}$, $w = \phi_\beta \circ \phi_\alpha^{-1}$, for K.

Example (continued)

- Let $M = \mathbb{P}^1(\mathbb{C})$ with the usual coordinate patches $U_0 = \{[z_0 : z_1] \in \mathbb{P}^1(\mathbb{C}),\ z_0 \neq 0\}$ and $U_1 = \{[z_0 : z_1] \in \mathbb{P}^1(\mathbb{C}),\ z_1 \neq 0\}$. Then the transition function $g_{01}(z) = z^n$ on $U_0 \cap U_1 = \mathbb{C}^\times$ defines a line bundle on $\mathbb{P}^1(\mathbb{C})$, usually denoted $\mathcal{O}(n)$. A section of this line bundle is given by holomorphic functions s_0 and s_1 on $\mathbb{C}$ such that

$$s_0(z) = z^n s_1(\tilde{z}),\ \tilde{z} = \frac{1}{z}$$

on the overlap. Expanding these functions in their respective local coordinates, and using $\tilde{z} = z^{-1}$ we obtain

$$\sum_{m \geqslant 0} a_m z^m = z^n \sum_{m \geqslant 0} \tilde{a_m} z^{-m}.$$

Example (continued)

Equating coefficients we get

$$\tilde{a_m} = a_m = 0,\ m > n$$

and

$$\tilde{a_0} = a_n,\ \tilde{a_1} = a_{n-1}, \ldots, \tilde{a_n} = a_0.$$

Thus the section is given by a polynomial of degree $\leqslant n$

$$\sum_0^n a_m z^m.$$

Hence the dimension of $H^0(\mathbb{P}^1(\mathbb{C}), \mathcal{O}(n)) = n + 1$.

Constructions (Operations on Line bundles)

- $L \mapsto L^{\star}$, *also denoted* L^{-1} *and called the dual of* L. *It has transition functions* $g_{\alpha\beta}(L^{\star}) = g_{\alpha\beta}^{-1}(L)$.
- $(L, \tilde{L}) \mapsto L \otimes \tilde{L}$. *It has transition functions*

$$g_{\alpha\beta}(L \otimes \tilde{L}) = g_{\alpha\beta}(L) g_{\alpha\beta}(\tilde{L}).$$

- $(L, \tilde{L}) \mapsto \mathrm{Hom}(L, \tilde{L}) \cong L^{\star} \otimes \tilde{L}$, *sections of* $\mathrm{Hom}(L, \tilde{L})$ *are holomorphic homomorphisms from* $L \to \tilde{L}$.
- $L \mapsto \mathrm{Hom}(L, L) \simeq L^{\star} \otimes L$ *is canonically trivial as the only endomorphisms of a one dimensional vector space are the scalars.*
- *If* s *is a section of* L *and* $\tilde{s}$ *a section of* $\tilde{L}$, *then the product* $s\tilde{s}$ *is a section of* $L \otimes \tilde{L}$, *usually just denoted* $L\tilde{L}$.

Definition

If M is a compact Riemann surface, its genus g_M is defined to be $\dim_{\mathbb{C}} H^0(M, K)$.

Example

$M = \mathbb{P}^1(\mathbb{C})$. A section of the canonical bundle looks like $f_0(z)dz$ on U_0 and $f_1(\tilde{z})d\tilde{z}$ on U_1 where f_0 and f_1 are holomorphic functions on $\mathbb{C}$. These forms must agree on the overlap $U_0 \cap U_1 = \mathbb{C}^\star$, $\tilde{z} = \frac{1}{z}$. This gives

$$d\tilde{z} = -z^{-2}dz, \ f_0(z)dz = -z^{-2}f_1(z^{-1})dz \Longrightarrow f_0(z) = f_1(\tilde{z}) = 0$$

$$\Longrightarrow g_M = 0.$$

Since $d\tilde{z} = z^{-2}(-dz)$, we have $K = \mathcal{O}(-2)$.

Remark

If a line bundle L has a nowhere vanishing section s then

$$M \times \mathbb{C} \to L,\ (m, u) \mapsto us(m)$$

is an isomorphism between L and the trivial line bundle.

Definition

A sheaf $\mathcal{S}$ on a topological space Y associates to each open set $U \subset Y$ an abelian group $\mathcal{S}(U)$ (sections over U) and to each inclusion $U \subset V \subset Y$ a restriction map $r_{VU} : \mathcal{S}(V) \to \mathcal{S}(U)$ such that

1. For $U \subset V \subset W$ we have
$$r_{WU} = r_{VU} \circ r_{WV}.$$
2. If $\sigma \in \mathcal{S}(U)$ and $\tau \in \mathcal{S}(V)$ are such that
$$r_{UU\cap V}(\sigma) = r_{VU\cap V}(\tau) \Longrightarrow \exists! \mu \in \mathcal{S}(U \cup V) \text{ with}$$
$$r_{U\cup VU}(\mu) = \sigma; \;\; r_{U\cup VV}(\mu) = \tau.$$
$$\mathcal{S}(\emptyset) = \{e\}, \, r_{UU} = \mathrm{Id}_U \, .$$

Example

- For a Riemann surface M, $\mathcal{S}(U) = \mathcal{O}_M(U)$: holomorphic functions on $U \subset M$.
- M a Riemann surface, L a line bundle over M, $\mathcal{S} = \mathcal{O}(L)$: elements of $\mathcal{O}(L)(U)$ are collections of functions h_i on $U \cap U_i$ such that $h_i = g_{ij}h_j$ on $U \cap U_i \cap U_j$. Here (U_i) are trivializing open sets and g_{ij} are the transition functions. If $U = M$ we recover the above definition of a section of a line bundle L over M.
- Y a topological space, $\mathcal{S}(U) =$ Locally constant functions on U with values on $\mathbb{C}$ or $\mathbb{Z}$.
- M a Riemann surface, $\mathcal{S}(U) = \mathcal{O}_M^\times(U) =$ nowhere vanishing functions on U, with the group operation being multiplicative.

If $\mathcal{S}$ is a sheaf on M, we can construct the Cech cohomology groups $H^p(M, \mathcal{S})$ with coefficients in $\mathcal{S}$ as follows. Take a (locally finite) covering $\{U_\alpha\}_{\alpha \in A}$ of M by open sets. Let

$$\mathcal{S}^0 = \oplus_\alpha \mathcal{S}(U_\alpha)$$

$$\mathcal{S}^1 = \oplus_{\alpha \neq \beta} \mathcal{S}(U_\alpha \cap U_\beta)$$

$$\vdots$$

$$\mathcal{S}^p = \oplus_{\alpha_0 \neq \ldots \neq \alpha_p} \mathcal{S}(U_{\alpha_0} \cap \ldots U_{\alpha_p})$$

and define $\mathcal{C}^p$ to be the alternating elements in $\mathcal{S}^p$. This means that for a permutation of the indices the open set does not change but one multiplies the section by the signature of the permutation.

Define the homomorphism of Abelian groups $\partial : \mathcal{C}^p \to \mathcal{C}^{p+1}$:

$$(\partial f)_{\alpha_0 \ldots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1^i f_{\alpha_0 \ldots \hat{\alpha_i} \ldots \alpha_{p+1}} \Big|_{U_{\alpha_0} \cap \ldots \cap U_{\alpha_{p+1}}} .$$

Here $f_{\alpha_0 \ldots \hat{\alpha_i} \ldots \alpha_{p+1}} \in \mathcal{S}(U_{\alpha_0} \cap \ldots U_{\alpha_{i-1}} \cap U_{\alpha_{i+1}} \ldots U_{\alpha_p})$. $\partial \leftrightarrow$ boundary operator and $\partial^2 = 0$. For instance if

$$f \in \mathcal{S}^0 = \mathcal{C}^0 \leftrightarrow f = (f_\alpha)_\alpha,\ f_\alpha \in \mathcal{S}(U_\alpha);$$

$$(\partial f)_{\alpha\beta} = f_\alpha \big|_{U_\alpha \cap U_\beta} - f_\beta \big|_{U_\alpha \cap U_\beta}.$$

Thus $\partial f = 0 \iff$ the local sections f_α piece together to give a global section.

Definition

The p-th cohomology group of M with coefficients in $\mathcal{S}$, relative to the covering $U = (U_\alpha)_\alpha$, is

$$H^p_U(M, \mathcal{S}) := \frac{\operatorname{Ker} \partial : \mathcal{C}^p \to \mathcal{C}^{p+1}}{\operatorname{Im} : \mathcal{C}^{p-1} \to \mathcal{C}^p}.$$

The p-th cohomology of M with coefficients in $\mathcal{S}$ is then the direct limit over the open covers U of M, partially ordered by refinement, of the $H^p_U(M, \mathcal{S})$:

$$H^p(M, \mathcal{S}) = \varinjlim_{\text{U covering of M}} H^p_U(M, \mathcal{S}).$$

Definition (continued)

In general this is incalculable. In order to get around this problem one chooses in practice a so-called good covering U, that is a covering for which the p-th cohomology, $p > 0$, of all the finite intersections of open sets of the covering, is zero:

$$H^p(U_{i_1} \cap \ldots \cap U_{i_k}, \mathcal{S}_{U_{i_1} \cap \ldots \cap U_{i_k}}) = 0, \ p > 0.$$

Then $H^p_U(M, \mathcal{S}) = H^p(M, \mathcal{S})$.

Remark

What happens is that if $V = (V_\beta)_\beta$ is a refinement of $U = (U_\alpha)_\alpha$: for any V_β, there exists α such that $V_\beta \subset U_\alpha$, a situation we write $U \leqslant V$, then we have a map

$$t_{UV} : H^p_U(M, \mathcal{S}) \to H^p_V(M, \mathcal{S})$$

and one shows that if $U \leqslant V \leqslant W$ then

$$t_{UW} = t_{VW} \circ t_{UV};$$

so (U, t_{UV}) is an inductive system and one can consider the direct limit.

Remark

If M is a Riemann surface, and we take a covering of M by connected open charts $U = \{U_\alpha\}_\alpha$, then we know that each finite intersection $U_{i_1} \cap \ldots \cap U_{i_k}$, if non-empty, is isomorphic to a non-empty connected non-compact open subset of $\mathbb{C}$. Let $\mathcal{O}_M$ be the sheaf of holomorphic functions on M, then we have

$$H^p(U_{i_1} \cap \ldots \cap U_{i_k}, \mathcal{O}_M\big|_{U_{i_1} \cap \ldots \cap U_{i_k}}) = 0, \ p > 0.$$

This follows from the Behnke-Stein theorem and Cartan Theorem B.

Example

For L a line bundle, with transition functions $g_{\alpha\beta} = \psi_\alpha \circ \psi_\beta^{-1}$, then $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$; so the family $g = (g_{\alpha\beta})$ lies in $\mathcal{C}^1$ for the sheaf $\mathcal{O}_M^\times$ of non-vanishing holomorphic functions. Furthermore

$$(\partial g)_{\alpha\beta\gamma} = g_{\beta\gamma} g_{\alpha\gamma}^{-1} g_{\alpha\beta} = \mathrm{Id}$$

so $g = (g_{\alpha\beta})_{\alpha\beta} \in \operatorname{Ker} \partial$. Hence it defines an element of $H^1(M, \mathcal{O}_M^\times)$. Actually one can show that the set of isomorphism classes of line bundles on M is $H^1(M, \mathcal{O}_M^\times)$.

Theorem (Vanishing theorem)

Let M be a Riemann surface. If $S = \mathcal{O}(L)$, the sheaf of holomorphic sections of the line bundle L then $H^p(M, \mathcal{S}) = 0$ for $p > 1$. If $\mathcal{S} = \mathbb{C}$ or $\mathbb{Z}$, then $H^p(M, \mathcal{S}) = 0$ for $p > 2$.

We set $H^1(M, \mathcal{O}(L)) = H^1(M, L)$.

Theorem (Serre duality)

If L is a line bundle on a compact Riemann surface M, then

$$H^1(M, L) = H^0(M, K \otimes L^\star)^\star.$$

Let us start with the exponential exact sequence on a compact Riemann surface M

$$0 \longrightarrow \mathbb{Z} \hookrightarrow \mathcal{O} \xrightarrow{e(2i\pi \cdot)} \mathcal{O}^{\times} \longrightarrow 1$$

where $e(2i\pi \cdot) : f \mapsto \exp(2i\pi f)$, f a section of $\mathcal{O}_M$.
By homological algebra this gives rise to a long exact sequence in cohomology

$$0 \longrightarrow \mathbb{Z} \hookrightarrow \mathbb{C} \longrightarrow \mathbb{C}^{\times} \longrightarrow H^1(M,\mathbb{Z}) \longrightarrow H^1(M,\mathcal{O}_M) \longrightarrow H^1(M,\mathcal{O}_M^{\times}) \longrightarrow H^2(M,\mathbb{Z}) \longrightarrow H^2(M,\mathcal{O}_M) \longrightarrow \dots$$

The first part of this sequence follows from the fact that holomorphic functions on a compact Riemann surface are constant. Since $\exp$ is surjective onto $\mathbb{C}^{\times}$, then from exactness we get the injection $H^1(M,\mathbb{Z}) \hookrightarrow H^1(M,\mathcal{O}_M)$. Also $H^2(M,\mathcal{O}_M) = 0$; so the previous long exact sequence reduces to the exact sequence

$$0 \longrightarrow \frac{H^1(M,\mathcal{O}_M)}{H^1(M,\mathbb{Z})} \hookrightarrow H^1(M,\mathcal{O}_M^{\times}) \xrightarrow{\delta} H^2(M,\mathbb{Z}) \longrightarrow 0$$

Since M is a two dimensional compact oriented connected manifold, we have by Poincaré duality

$$H^2(M,\mathbb{Z}) \simeq \mathbb{Z}.$$

Definition

The degree of the line bundle L is $\delta([L])$. It is denoted $\deg L$. The degree depends only on the isomorphism class $[L]$, the class of L in $H^1(M, \mathcal{O}_M^\times)$.

Properties

We have the following properties

- *For* L, $\tilde{L}$, 2 *line bundles, then*

$$\deg(L \otimes \tilde{L}) = \deg(L) + \deg(\tilde{L}),$$

  (δ *is a homomorphism).*

- $\deg L_p = 1.$
- *If* $s \in H^0(M, L)$ *vanishes only at the points* $p_1, \ldots, p_n$ *with multiplicities* $m_1, \ldots, m_n$ *then*

$$\deg(L) = \sum_i m_i.$$

To see this take

$$ss_{p_1}^{-m_1}\dots s_{p_n}^{-m_n}$$

which is a non-vanishing section of $LL_{p_1}^{-m_1}\dots L_{p_1}^{-m_n}$. So $LL_{p_1}^{-m_1}\dots L_{p_1}^{-m_n}$ is trivial and this gives

$$L \simeq L_{p_1}^{m_1}\dots L_{p_1}^{m_n}.$$

Corollary

If $\deg L < 0$, then L has no non-trivial holomorphic sections.

Definition

A rank r vector bundle over a Riemann surface M is a complex manifold E of dimension $r+1$ with a holomorphic projection $\pi : E \to M$ such that

- for each $z \in M$, $\pi^{-1}(z)$ is an r-dimensional complex vector space.
- each point $m \in M$ has a neighborhood U and a homeomorphism ψ_U such that

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\psi_U} & U \times \mathbb{C}^r \\ \downarrow \pi & \swarrow pr_U & \\ U & & \end{array}$$

  is commutative.

Definition (continued)

- The transition functions $\psi_V \circ \psi_U^{-1}$ are of the form

$$(z, w) \mapsto (z, A(z)w)$$

where $A : U \cap V \to \mathrm{GL(r, \mathbb{C})}$ is a holomorphic map to the space of invertible $r \times r$ matrices.

Remark

As with line bundles, if we denote $A \leftrightarrow g_{VU}$ then

$$g_{UV} g_{VW} = g_{UW}.$$

Constructions

- $(E, \tilde{E}) \mapsto E \oplus \tilde{E}$, *their direct sum with rank* $\mathrm{rank}(E \oplus \tilde{E}) = \mathrm{rank}(E) + \mathrm{rank}(\tilde{E})$.
- $(E, \tilde{E}) \mapsto E \otimes \tilde{E}$, *their tensor product with rank*

$$\mathrm{rank}(E \otimes \tilde{E}) = \mathrm{rank}\, E \,\mathrm{rank}\, \tilde{E}.$$

- $E \mapsto E^{\star}$, *the dual of* E *with the same rank as* E.
- $E \mapsto \det(E) := \wedge^{\mathrm{rank}\, E}(E)$, *a line bundle with transition functions* $\det(g_{\alpha\beta})$.

Definition

If E is a vector bundle on M, we define its degree by

$$\deg(E) := \deg(\det(E))$$

the degree of the line bundle $\det(E)$.

Theorem (Vanishing theorem for vector bundles)

If $\mathcal{O}(E)$ is the sheaf of holomorphic sections of a vector bundle E on the compact Riemann surface M. Then

$$H^p(M, \mathcal{O}(E)) = 0, \ p > 1.$$

Theorem (Serre duality for vector bundles on Riemann surfaces)

If M is a compact Riemann surface we have

$$H^1(M, \mathcal{O}(E)) \simeq H^0(M, K \otimes E^\star)^\star.$$

We also have the

Theorem (Riemann-Roch)

If E is a vector bundle on a compact Riemann surface M of genus g_M, then

$$\dim H^0(M, E) - \dim H^1(M, E) = \deg E + \operatorname{rank}(E) \times (1 - g_M).$$

Theorem (Birkhoff-Grothendieck)

If E is a rank r holomorphic vector bundle on $\mathbb{P}^1(\mathbb{C})$, then

$$E \cong \mathcal{O}(a_1) \oplus \ldots \mathcal{O}(a_r)$$

for some $a_i \in \mathbb{Z}$. Furthermore
$\mathcal{O}(a_1) \oplus \ldots \oplus \mathcal{O}(a_r) \cong \mathcal{O}(b_1) \oplus \ldots \oplus \mathcal{O}(b'_r)$ if and only if $r = r'$ and up to reordering $a_i = b'_i$.[a]

[a] Grothendieck, "Sur la classification des fibrés holomorphes sur la sphère de Riemann".

Proof.

Idea of proof: apply induction on the rank of the vector bundle E. One shows that for large $n >> 0$, $E(n) := E \otimes \mathcal{O}(n)$

Proof (cont.)

splits as

$$E(n) = \mathcal{O} \oplus \mathcal{Q}$$

where $\mathcal{Q}$ is a vector bundle of rank $r-1$. In the proof of the splitting of $E(n)$ at crucial points one applies the Riemann-Roch theorem. By induction

$$\mathcal{Q} \cong \mathcal{O}(b_1) \oplus \ldots \oplus \mathcal{O}(b_{r-1}).$$

Hence

$$E \cong \mathcal{O}(-n) \oplus \mathcal{O}(b_1 - n) \oplus \ldots \oplus \mathcal{O}(b_{r-1} - n).$$

□

Corollary

Let E be a holomorphic vector bundle of rank r over $\mathbb{P}^1(\mathbb{C})$. Then E is trivial $\iff \deg(E) = 0$ and $H^0(\mathbb{P}^1(\mathbb{C}), E(-1)) = 0$.

Why does the name of Birkhoff appears in the theorem? Birkhoff was interested in the Riemann Hilbert problem: given a linear homogeneous differential on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, say, any basis of solution of it can be analytically continued along any closed path of $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ and remains a basis after analytic continuation. This process depends only on the homotopy class of the curve in question, and defines a representation of the fundamental group of $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ into $\mathrm{GL(r, \mathbb{C})}$, where r is the order of the differential equation. One can invert this problem and ask for the existence of a differential equation with prescribed monodromy representation. This amounts to the construction of a vector bundle on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ with flat connection.

Definition

Let $E \xrightarrow{\pi} M$ be a holomorphic vector bundle of rank r on a Riemann surface M. A holomorphic subbundle $F \subseteq E$ is a collection of subspaces $\{F_x \subset E_x\}_{x \in M}$ of the fibers $E_x = \pi^{-1}(x)$ such that $F = \cup_{x \in M} F_x$ is a submanifold of E; this means that every $x \in M$ has a neighborhood U and a trivialization

$$\psi_U : E_U \to U \times \mathbb{C}^r, \ E_U := \pi^{-1}(U)$$

such that

$$\psi_U\big|_{F_U} : F_U \to U \times \mathbb{C}^s \subset U \times \mathbb{C}^r, \ F_U := (\pi \circ i)^{-1}(U),$$

where $i\big|_{F_x}$ is the inclusion of F_x into E_x.

Definition

A non-zero holomorphic vector bundle on a connected complex manifold is indecomposable if it is not the direct sum of two non-zero subbundles.

Theorem (Atiyah-Krull-Remak-Schmidt)

Any holomorphic vector bundle on a connected compact complex manifold M is a direct sum of indecomposable subbundles. Furthermore let $E_1, \ldots, E_m$ and $F_1, \ldots, F_n$ be indecomposable holomorphic vector bundles on M, such that $E_1 \oplus \ldots \oplus E_m$ is isomorphic to $F_1 \oplus \ldots \oplus F_n$. Then $m = n$, and up to permutation of indices, $E_1, \ldots, E_m$ are isomorphic to $F_1, \ldots, F_n$.[a]

[a] Atiyah, "On the Krull-Schmidt theorem with application to sheaves".

Remark

Let $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M$ be two holomorphic vector bundles on the complex manifold M. Then $\varphi : E \to E'$ is a homomorphism if and only if

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & E' \\ \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle \pi'} & \\ M & & \end{array}$$

is commutative.

Further reading

The theory of vector bundles on compact Riemann surfaces and more generally of Moduli spaces of vector bundles on compact Riemann surfaces has developed tremendously since the trailblazing works of Grothendieck. Prior to the work of Grothendieck we had very important works of Appell[a], Humbert[b], which classify line bundles on complex toruses $X = \mathbb{C}^g/\Lambda$, $g \geqslant 1$, Λ a lattice of $\mathbb{C}^g$, by the so-called factors of automorphy

$$H^1(X, \mathcal{O}_X^\times) = H^1(\pi_1(X), H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}^\times)).$$

[a] P. E. Appell. "Sur les fonctions périodiques de deux variables". In: *Journ. de Math. Pures Appli.* (4) VII (1891), pp. 157–219.

[b] G. M. Humbert. "Théorie générale des surfaces hyperelliptiques". In: *Journ. de Math. Pures Appli.* (4) IX (1893), pp. 29–170, 361–475.

Further reading (continued)

We also had the work of Weil[a], which classified vector bundles on compact Riemann surfaces M of genus $g \geqslant 2$, say, which arise from representations of the fundamental group of the surface. According to it such a vector bundle arises from a representation of the fundamental group if and only if each of its indecomposable subbundles, are of degree 0.

[a] A. Weil. "Généralisation des fonctions abéliennes". In: *J. Math. Pures Appl.* (9) 17 (1938), pp. 47–87.

Further reading (continued)

Most modern proofs of the theorem of Weil use the theorem of Atiyah[a] on the existence of complex analytic connections on a holomorphic vector bundle over a complex manifold, M. Let $E \xrightarrow{\pi} M$ be a vector bundle of rank $r \geqslant 1$ over M, with local trivializations $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^r$, and transitions functions $g_{\beta\alpha} = \psi_\beta \circ \psi_\alpha^{-1}$. Then E admits a complex analytic connection if and only if

$$\{(U_{\alpha\beta} = U_\alpha \cap U_\beta, \psi_\beta^{-1}(dg_{\beta\alpha})(g_{\beta\alpha}^{-1})\psi_\beta)\}$$

which defines a class in $H^1(M, \Omega^1_M \otimes \mathrm{End}(E))$, vanishes.

[a] Atiyah, "Complex analytic connections in fibre bundles".

Further reading (continued)

For more on the moduli aspects of the theory we refer to Narasimhan-Seshadri[a], Hitchin[b] and Kobayashi[c], and references thereien.

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[a] C. S. Seshadri M. S. Narasimhan. "Stable and unitary bundles on a compact Riemann surface". In: *Ann. Math. (2)* 82 (1965), pp. 540–567.

[b] N. J. Hitchin. "Stable bundles and integrable systems". In: *Duke Math. J.* 54 (1987), pp. 91–114.

[c] S. Kobayashi. *Differential geometry of complex vector bundles*. Kanô Memorial Lectures, 5. Publications of the Mathematical Society of Japan, 15, 1986.

References

Appell, P. E. "Sur les fonctions périodiques de deux variables". In: *Journ. de Math. Pures Appli.* (4) VII (1891), pp. 157–219.

Atiyah, M. F. "Complex analytic connections in fibre bundles". In: *Trans. Am. Math. Soc.* 85 (1957), pp. 181–207.

– ."On the Krull-Schmidt theorem with application to sheaves". In: *Bull. Soc. Math. Fra.* 84 (1956), pp. 307–317.

– ."Vector bundles over an elliptic curve". In: *Proc. Lond. Math. Soc.* (3) 7 (1957), pp. 414–452.

Grothendieck, A. "Sur la classification des fibrés holomorphes sur la sphère de Riemann". In: *Am. J. Math.* 79 (1957), pp. 121–138.

Hitchin, N. J. *Integrable systems, twistors, loop groups and Riemann Surfaces*. Oxford University Press, 1999, pp. 11–52.

References (cont.)

Hitchin, N. J. "Stable bundles and integrable systems". In: *Duke Math. J.* 54 (1987), pp. 91–114.

Humbert, G. M. "Théorie générale des surfaces hyperelliptiques". In: *Journ. de Math. Pures Appli.* (4) IX (1893), pp. 29–170, 361–475.

Kobayashi, S. *Differential geometry of complex vector bundles*. Kanô Memorial Lectures, 5. Publications of the Mathematical Society of Japan, 15, 1986.

M. S. Narasimhan, C. S. Seshadri. "Stable and unitary bundles on a compact Riemann surface". In: *Ann. Math. (2)* 82 (1965), pp. 540–567.

Weil, A. "Généralisation des fonctions abéliennes". In: *J. Math. Pures Appl.* (9) 17 (1938), pp. 47–87.

Thank you for your attention.