Superoscillations, the Talbot Carpet, and Gauss Sums

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Joint work with Fabrizio Colombo, Irene Sabadini, and Alain Yger
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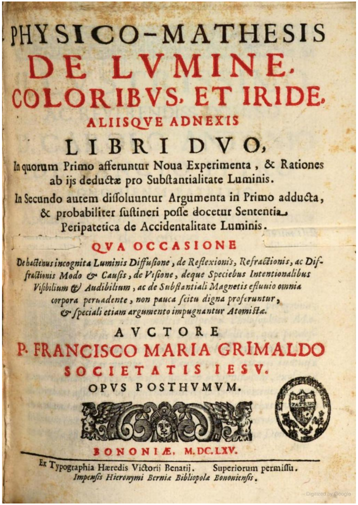
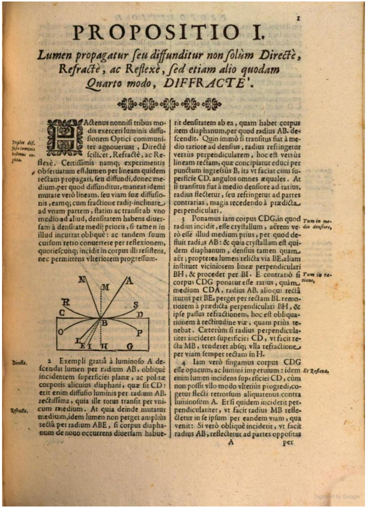
The Talbot Carpet



PHYSICO-MATHESIS
DE LVMINE,
COLORIBVS, ET IRIDE,
ALIISQVE ADNEXIS
LIBRI DVO,
In quorum Primo afferuntur Noua Experimenta, & Rationes ab ijs deductæ pro Substantialitate Luminis.
In Secundo autem dissoluuntur Argumenta in Primo adducta, & probabiliter sustineri posse docetur Sententia Peripatetica de Accidentalitate Luminis.
QVA OCCASIONE
De hactenus incognita Luminis Diffusione, de Reflexionis, Refractionis, ac Diffractionis Modo & Causis, de Visione, deque Speciebus Intentionalibus Visibilium & Audibilium, ac de Substantiali Magnetis effluuio omnia corpora peruadente, non pauca scitu digna proferuntur, & speciali etiam argumento impugnantur Atomistæ.
AVCTORE
P. FRANCISCO MARIA GRIMALDO
SOCIETATIS IESV.
OPVS POSTHVMVM.
BONONIÆ, M.DC.LXV.
Ex Typographia Hæredis Victorij Benatij. Superiorum permissu.
Impensis Hieronymi Berniæ Bibliopolæ Bononiensis.

The Talbot Carpet



1

PROPOSITIO I.

Lumen propagatur seu diffunditur non solùm Directè, Refractè, ac Reflexè, sed etiam alio quodam Quarto modo, DIFFRACTE'.

Triplex diffusio luminis hactenus cognita.

HActenus nonnisi tribus modis exerceri luminis diffusionem Optici communiter agnouerunt, Directè scilicet, Refractè, ac Reflexè. Certissimis namq; experimentis obseruatum est, lumen per lineam quidem rectam propagari, seu diffundi, donec medium, per quod diffunditur, maneat idem: mutare verò lineam, seu viam suæ diffusionis, eamq; cum fractione radij inclinare ad vnam partem, statim ac transit ab vno medio ad aliud, densitatem habens diuersam à densitate medij prioris, si tamen in illud incurrat obliquè: ac tandem suum cursum retro conuertere per reflexionem, quotiescunq; incidit in corpus illi resistens, nec permittens vlteriorem progressum.

Directa.

2 Exempli gratiâ à luminoso A descendat lumen per radium AB, obliquè incidentem superficiei planæ, ac politæ corporis alicuius diaphani, quæ sit CD: erit enim diffusio luminis per radium AB. rectissima, quia ille totus transit per vnicum medium. At quia deinde mutatur *Refracta.* medium, idem lumen non perget ampliùs rectâ per radium ABE, si corpus diaphanum de nouo occurrens diuersam habuerit densitatem ab ea, quam habet corpus item diaphanum, per quod radius AB. descendit. Quin immò si transitus fiat à medio rariore ad densius, radius refringetur versùs perpendicularem, hoc est versùs lineam rectam, quæ concipiatur educi per punctum ingressûs B, ita vt faciat cum superficie CD. angulos omnes æquales. At si transitus fiat à medio densiore ad rarius, radius flectetur, seu refringetur ad partes contrarias, magis recedendo à prædicta perpendiculari.

3 Ponamus iam corpus CDG, in quod radius incidit, esse crystallum, aërem verò esse illud medium prius, per quod defluit radius AB: & quia crystallum est quidem diaphanum, densius tamen quam aër; propterea lumen relicta via BE, aliam instituet viciniorem lineæ perpendiculari BH, & procedet per BI. E contrariò si corpus CDG ponatur esse rarius, quàm medium CDA, radius AB. alioqui rectâ iturus per BE, perget per rectam BL remotiorem à prædicta perpendiculari BH, & ipse passus refractionem, hoc est obliquationem à rectitudine viæ, quam priùs tenebat. Cæterùm si radius perpendiculariter incideret superficiei CD, vt facit recta MB, tenderet absq; vlla refractione per viam semper rectam in H.

Tum in medio densiore,

Tum in ratione,

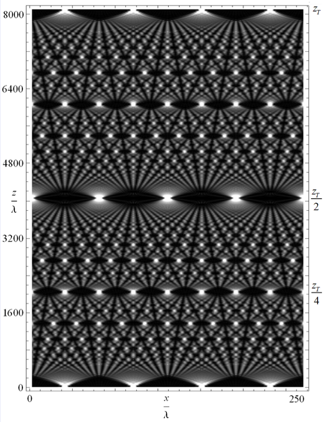
Et Reflexa.

4 Iam verò fingamus corpus CDG esse opacum, ac lumini imperuium: idem enim lumen incidens superficiei CD, cùm non possit vllo modo vlteriùs progredi, cogetur flecti retrorsum aliquatenus contra luminosum A. Et si quidem inciderit perpendiculariter, vt facit radius MB reflectetur in se ipsum per eandem viam, qua venit: Si verò obliquè inciderit, vt facit radius AB, reflectetur ad partes oppositas

A per

The Talbot Carpet

H.F.Talbot, The London and Edinburgh Philosophical Magazine and Journal of Science, December 1836



The Talbot Carpet

Diffraction effect caused by plane waves incident to a periodic diffraction grating

Image of grating is repeated at what is known as the *Talbot Length*. The quantity depends on the period a and the wavelength λ, and (Lord Rayleigh) is

$$T = \frac{\lambda}{1 - \sqrt{1 - \frac{\lambda^2}{a^2}}} \approx \frac{2a^2}{\lambda} \text{ if} \lambda << a$$

At half length we see the same image, with half a period shift

At quarter length we see the same image with half a period and half size, etc. etc,

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Gauss Sums: an Excursus in Number Theory

Normal Quadratic Gauss Sums

For a, c coprime numbers, the normal quadratic Gauss sums are defined by

$$G(a,c) := \sum_{\ell=0}^{c-1} e^{2\pi i \frac{a\ell^2}{c}}$$

C.F.Gauss, *Disquisitiones Arithmeticae*, 1801.

The Generalized Quadratic Gauss Sums

$$G(a,b,c) = \sum_{\ell=0}^{c-1} e^{2\pi i \frac{a\ell^2+b\ell}{c}}$$

$$G(a,c) = G(a,0,c)$$

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Superoscillating Functions: The Basics

"Superoscillating functions are bandlimited functions which can oscillate faster than the highest frequency that they contain."

They have originated in some works of Y. Aharonov and co-authors in the context of quantum mechanics (weak measurements) and further studied by M. Berry and co-authors.
Y. Aharonov, D. Albert, L. Vaidman, *How the result of a measurement of a component of the spin of a spin-1/2 particle can turn out to be 100*, Phys. Rev. Lett., (1988)
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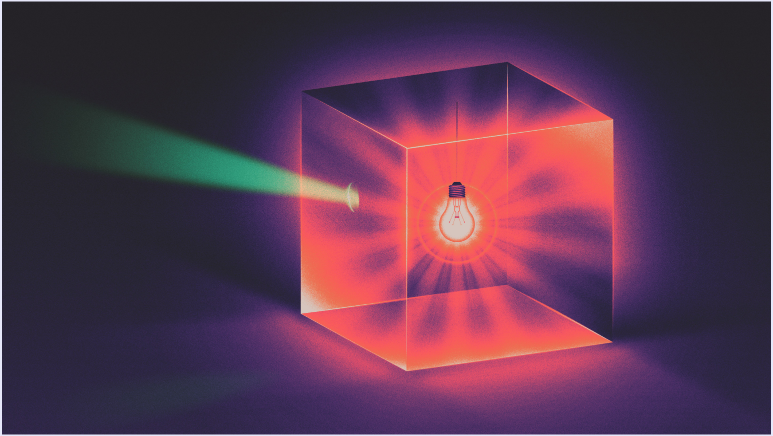
Superoscillating Functions: The Basics

Quanta Magazine, May 2022



Superoscillating Functions: The Basics

Prototypical Example of Superoscillating Function

$$F_n(x,a) := \left(\cos\left(\frac{x}{n}\right) + ia\sin\left(\frac{x}{n}\right)\right)^n,$$

where $a > 1$, $n \in \mathbb{N}$.

Superoscillating Functions: The Basics

Proposition

$F_n(x,a)$ can be written in terms of its Fourier coefficients $C_j(n,a)$ as

$$F_n(x,a) = \sum_{j=0}^{n} C_j(n,a) e^{i(1-2j/n)x},$$

where $C_j(n,a) := \frac{(-1)^j}{2^n}\binom{n}{j}(a+1)^{n-j}(a-1)^j$.

Superoscillating Functions: The Basics

Theorem

Let $F_n(x, a)$ be as above. Then for every $x \in \mathbb{R}$ we have

$$\lim_{n\to\infty} F_n(x, a) = e^{iax},$$

and the convergence is uniform on the compact sets in $\mathbb{R}$.

Remark

The term *superoscillating* comes from the fact that the frequencies satisfy $|1 - 2\frac{j}{n}| \leq 1$, however $F_n(x, a) \to e^{iax}$, $a > 1$.

Evolution of Superoscillations

Question:

Consider the Cauchy problem

$$i\frac{\partial\psi(x,t)}{\partial t} = H\psi(x,t), \qquad \psi(x,0) = F_n(x,a),$$

where

$$H\psi(x,t) := \left[-\frac{\partial^2}{\partial x^2} + V(x,t)\right]\psi(x,t).$$

How do supoeroscillations evolve?

Evolution of Superoscillations

General strategy:

- We consider the Cauchy problem

$$i\partial_t\psi(t,x) = H\psi(t,x), \quad \psi(0,x) = F_n(x,a)$$

  where H is the Hamiltonian operator of some physical system.
- We determine the solution $\psi_n(t,x;a)$ of the Cauchy problem using the Green function associated to the Hamiltonian H
- We show that the solution can be written as $\psi_n(t,x;a) = U(t, \frac{d}{dx})(F_n(x;a))$ for U a suitable infinite order/convolution operator
- We show that U acts continuously on the appropriate space of functions, and use this to calculate the limit of ψ_n and to prove that the superoscillatory behavior persists.

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Theorem

The solution of the Cauchy problem

$$i\frac{\partial\psi(x,t)}{\partial t} = \frac{\partial^2\psi(x,t)}{\partial x^2} \quad \psi(x,0) = F_n(x,a),$$

is given by

$$\psi_n(x,t) = \sum_{k=0}^{n} C_k(n,a)e^{ik_j(n)x}e^{-itk_j(n)^2}.$$

Proof: by inspection.

Evolution of Superoscillations

Theorem

The function

$$\psi_n(x,t) = \sum_{k=0}^{n} C_k(n,a) e^{ik_j(n)x} e^{-itk_j(n)^2}. \quad (1)$$

can be written as

$$\psi_n(x,t) = \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \frac{d^{2m}}{dx^{2m}} F_n(x)$$

for every $x \in \mathbb{R}$ and $t \in \mathbb{R}$.

Evolution of Superoscillations

Set

$$U(x,t) := \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \frac{d^{2m}}{dx^{2m}};$$

we have to prove that $U(x,t)$ acts continuously on a function space that contains $F_n(x,a)$, so that

$$\lim_{n\to\infty} \psi_n(x,t) = \lim_{n\to\infty} U(x,t)F_n(x,a) =$$

$$= U(x,t) \lim_{n\to\infty} F_n(x,a) = U(x,t)e^{iax} = e^{iax}e^{-ia^2 t}.$$

Evolution of Superoscillations

Problem

Continuity, on suitable function spaces, of the operator

$$U(z,t) := \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \frac{d^{2m}}{dz^{2m}}$$

Answer

We now have a full description of continuous linear operators on these spaces. In particular the operator $U(x,t)$ of the previous slide is indeed continuous.

Distribution Spaces

The convergence of $\{F_n(x;a)\}$ towards e^{iax} is uniform on any compact set of $\mathbb{R}$, and thus it holds in $\mathcal{D}'(\mathbb{R},\mathbb{C})$, but not in $\mathcal{S}'(\mathbb{R},\mathbb{C})$.

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Evolution of Superoscillations

The Dirac Comb and the Poisson Summation Formula

$$u_M(x) := \frac{2\pi}{M} \sum_{k\in\mathbb{Z}} \delta\left(x - \frac{2k\pi}{M}\right) = \sum_{k\in\mathbb{Z}} e^{iMkx}.$$

Next Step

Evolve the Dirac Comb, thus modeling a periodic grating along a vertical axis.

Evolution of Superoscillations

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Evolution of Superoscillations

The Result of the Evolution

The initial value given by the Dirac comb $u_M(x) = \frac{2\pi}{M} \sum_{k\in\mathbb{Z}} \delta(x - \frac{2k\pi}{M})$ evolves, in $\mathcal{D}'(\mathbb{R}_t^+ \times \mathbb{R}_x, \mathbb{C})$, as

$$\varphi_M(t,x) = \sum_{k\in\mathbb{Z}} e^{-i(Mk)^2 t} e^{iMkx}$$

The Arrival of the Gauss Sums

Let $q \in (N)^*$, $p \in \{0, \dots, q-1\}$ coprime with q and $t_{M,p,q} = \frac{2\pi p}{M^2 q}$. Then, in $\mathcal{D}'$, $\varphi_M(t,x)_{|\{t_{M,p,q}\times\mathbb{R}\}} =$

$$= \delta(t - t_{M,p,q}) \otimes \sum_{j=0}^{q-1} G(-p,j,q)(\frac{2\pi}{Mq} \sum_{k\in\mathbb{Z}} \delta(x - \frac{2k\pi q - 2\pi j}{Mq})).$$

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Evolution of Superoscillations

The Talbot carpet is a way to optically recover the values of the generalized Gauss functions. Moreover, such sums support what is experimentally visible, specifically the vanishing of $G(-p, j, q)$ when $q = 2q'$, $q' - j \equiv 1$ modulo 2.

$t = \frac{1}{2}\frac{2\pi}{M^2}$, $p = 1$, $q = 2$, $j = 0, 1$

$$G(-1,0,2)\left(\frac{2\pi}{2M}\sum_{\mathbb{Z}}\delta\left(x - \frac{2k\pi}{M}\right)\right) + G(-1,1,2)\left(\frac{2\pi}{2M}\sum_{\mathbb{Z}}\delta\left(x - \frac{2k\pi}{M} - \frac{2\pi}{2M}\right)\right) =$$

$$= G(-1,1,2)\left(\frac{2\pi}{2M}\sum_{\mathbb{Z}}\delta\left(x - \frac{2k\pi}{M} - \frac{2\pi}{2M}\right)\right)$$

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$$G(-1,0,2)\left(\frac{2\pi}{2M}\sum_{\mathbb{Z}}\delta\left(x-\frac{2k\pi}{M}\right)\right)+G(-1,1,2)\left(\frac{2\pi}{2M}\sum_{\mathbb{Z}}\delta\left(x-\frac{2k\pi}{M}-\frac{2\pi}{2M}\right)\right)=$$

$$= G(-1,1,2)\left(\frac{2\pi}{2M}\sum_{\mathbb{Z}}\delta\left(x-\frac{2k\pi}{M}-\frac{2\pi}{2M}\right)\right)$$

Superoscillations and Gauss Sums

Key point is that exponentials of arbitrarily large frequencies can be approximated uniformly by band-limited exponentials. This opens the way to recuperate the values of the Gauss sums asymptotically from the values of the Fourier Transform of a band-limited function.

Superoscillations and Gauss Sums

The Evolution of the Regularized Dirac Comb

Let $\psi \in C^2(\mathbb{R}, \mathbb{C})$ with compact support. The regularized Dirac comb

$$x \mapsto (u_M * \psi)(x) = \Big[\sum_{k\in\mathbb{Z}} e^{ikMx} * \psi\Big] = \Big[\sum_{k\in\mathbb{Z}} e^{ikMx}\widehat{\psi}(kM)\Big] \qquad (2)$$

evolves to $(t, x) \mapsto \varphi_M(t, x) * \psi(x)$ in $\mathscr{D}'(\mathbb{R}^+_t \times \mathbb{R}_x, \mathbb{C})$, where φ_M is the evolution of the Dirac comb.

Superoscillations and Gauss Sums

Recall Result on Talbot Carpet

$$\varphi_M(t,x)_{|\{t_{M,p,q}\times\mathbb{R}\}} =$$

$$= \delta(t - t_{M,p,q}) \otimes \sum_{j=0}^{q-1} G(-p,j,q)\left(\frac{2\pi}{Mq}\sum_{k\in\mathbb{Z}} \delta(x - \frac{2k\pi q - 2\pi j}{Mq}\right)).$$

Corollary

With the same notations used up to now, $supp(\psi) \subset [-1,1]$, and $\psi(0) = 1$ we obtain

$$G(-p,j,q) = \frac{Mq}{2\pi} < \varphi_M(t,x), \psi(\frac{Mq}{2\pi}(x - \frac{2j\pi}{Mq})) >$$

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Superoscillations and Gauss Sums

Given $N, N' \in \mathbb{N}^*$, $\nu \in \{0, ..., N-1\}$, $\nu' \in \{0, ..., N'-1\}$ and $\kappa \in \mathbb{N}$, let

$$\omega_{\nu,\nu'}^{N,N'}(\kappa) := \exp\Big(-2i\pi\Big(\frac{1}{2}-\frac{\nu}{N}\Big)\Big(\kappa+\Big(\frac{1}{2}-\frac{\nu'}{N'}\Big)\Big)\Big). \quad (3)$$

Also set

$$C = C_{k,\nu,\nu'}^{p,q,N_K,N_{K'}} := \Big(\tfrac{1}{2}+\tfrac{k}{q}\Big)^{N_K-\nu}\Big(\tfrac{1}{2}-\tfrac{k}{q}\Big)^{\nu}\Big(\tfrac{1}{2}-kp\Big)^{N'_K-\nu'}\Big(\tfrac{1}{2}+kp\Big)^{\nu'}$$

Superoscillations and Gauss Sums

The Final Result

Let $(N_K)_{K\geq 1}$ and $(N'_K)_{K\geq 1}$ be two sequences of strictly positive integers such that $\lim_{K\to+\infty} \frac{\log N_K}{K} = \lim_{K\to+\infty} \frac{\log N'_K}{K} = +\infty$. Then, for any $q \in \mathbb{N}^*$, $\kappa \in \{0, \ldots, q-1\}$, $p \in \{1, \ldots, q-1\}$ coprime with q, and $\psi \in \mathcal{C}^2(\mathbb{R}, \mathbb{C})$ with compact support in $[-1, 1]$ such that $\psi(0) = 1$, there are positive constants $C = C_{k,\nu,\nu'}^{p,q,N_K,N_{K'}}$ such that

$$G(-p, \kappa, q) = \lim_{K\to+\infty} \sum_{k=-K}^{K} \sum_{\nu=0}^{N_K} \sum_{\nu'=0}^{N'_K} C\omega_{\nu,\nu'}^{N_K,N'_K}(\kappa)\, \widehat{\psi}\Big(2\pi\Big(\frac{1}{2} - \frac{\nu}{N_K}\Big)\Big).$$

References

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