The Isomorphism Theorem of Algebraic Logic: a Categorical Perspective

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  Syntactic consequence and semantic consequence coincide:

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- Accordingly, we could rewrite the Completeness Theorem:

$$\Sigma \vdash_{\mathrm{CL}} \varphi \quad \Leftrightarrow \quad \{\psi \approx 1 : \psi \in \Sigma\} \vDash_{\mathbf{2}} \varphi \approx 1.$$

Abstract deductive relations

An abstract deductive relation (ADR) on a set S is a relation $\vdash \subseteq \mathcal{P}(S) \times S$ such that, for every $X \subseteq S$ and $\varphi \in S$,

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3. if $X \subseteq Y$ and $X \vdash \varphi$, then $Y \vdash \varphi$. (It follows from 1 and 2.)

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A closure operator on a set S is a map $\gamma : \mathcal{P}S \to \mathcal{P}S$:

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These three concepts encode the same information. In particular, the closure system associated to an ADR $\vdash$ *is the set of its theories,* $\mathrm{Th}(\vdash)$.

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A theory of a logic $\vdash$ on S is a subset X so that if $X \vdash \varphi$ then $\varphi \in X$.

$\mathbf{Th}(\vdash) = \langle \mathrm{Th}(\vdash), \subseteq \rangle$ is a complete lattice.

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substitution-invariance: for all Σ, φ, and $\sigma\colon \mathbf{Fm}_{\mathcal{L}} \to \mathbf{Fm}_{\mathcal{L}}$,

$$\Sigma \vdash_{\mathcal{S}} \varphi \quad \Rightarrow \quad \sigma[\Sigma] \vdash_{\mathcal{S}} \sigma(\varphi).$$

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Sentential logics can be defined in multiple ways:

- via a deductive system (natural, Hilbert system, Gentsen system, . . .)
- via a semantics (algebraic, tableaux, model, . . .)
- via an abstract description (the smallest one satisfying this and this, . . .)
- . . .

Example 2: Equational logics

Let K be a class of algebras, Π a set of equations, and $\varepsilon_1 \approx \varepsilon_2$ an equation.

$$\Pi \vDash_{\mathsf{K}} \varepsilon_1 \approx \varepsilon_2$$

means that for every algebra $\mathbf{A} \in \mathsf{K}$ and every valuation $v \colon \mathbf{Fm} \to \mathbf{A}$,

if v satisfies all the equations of Π, then v also satisfies $\varepsilon_1 \approx \varepsilon_2$.

- $\vDash_{\mathsf{K}}$ is the equational logic associated to K.
- $\vDash_{\mathsf{K}}$ is an ADR on the set of equations $\mathrm{Eq}_{\mathcal{L}}$.
- $\vDash_{\mathsf{K}}$ also satisfies

substitution-invariance: for all Π, $\varepsilon_1 \approx \varepsilon_2$, and $\sigma \colon \mathbf{Fm}_{\mathcal{L}} \to \mathbf{Fm}_{\mathcal{L}}$,

$$\Pi \vDash_{\mathsf{K}} \varepsilon_1 \approx \varepsilon_2 \quad \Rightarrow \quad \sigma[\Pi] \vDash_{\mathsf{K}} \sigma(\varepsilon_1) \approx \sigma(\varepsilon_2).$$

Algebraic semantics

Given a set of equations Ξ in one variable x, we define the translation $\tau\colon \mathcal{P}(\mathrm{Fm}) \to \mathcal{P}(\mathrm{Eq})$ as follows:

$$\tau\{\psi\} = \Xi(\psi/x) \qquad \text{and} \qquad \tau\Sigma = \bigcup_{\psi\in\Sigma} \tau\{\psi\}.$$

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Definition

A class of algebras K is an algebraic semantics for a sentential logic $\mathcal{S}$ if there exists a set of equations Ξ in one variable x such that for every set of formulas Σ and every formula φ,

$$\Sigma \vdash_{\mathcal{S}} \varphi \quad \Leftrightarrow \quad \tau\Sigma \vDash_{\mathsf{K}} \tau\{\varphi\}.$$

Ξ is called a set of defining equations.

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so that for any set of formulas $\Sigma \cup \{\varphi\}$ and set of equations $\Pi \cup \{\varepsilon_1 \approx \varepsilon_2\}$

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Conditions 3 and 4 follow from 1 and 2.

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- The class HA is not an equivalent algebraic semantics for CL.

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- (M1) $1 \cdot \varphi = \varphi$.
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An ADR $\vdash$ on S is substitution-invariant if

$$X \vdash \varphi \quad \Rightarrow \quad \sigma \cdot X \vdash \sigma \cdot \varphi.$$

Equivalences of logics

Translation equivalence

Two ADR's $\vdash$ on S and $\vdash'$ on S' are translation equivalent if there exist maps $\tau\colon \mathcal{P}S \to \mathcal{P}S'$ and $\rho\colon \mathcal{P}S' \to \mathcal{P}S$ such that:

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- $X \dashv\vdash \rho\tau X$,
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Equivalences of logics

Translation equivalence

Two ADR's $\vdash$ on S and $\vdash'$ on S' are translation equivalent if there exist maps $\tau\colon \mathcal{P}S \to \mathcal{P}S'$ and $\rho\colon \mathcal{P}S' \to \mathcal{P}S$ such that:

- τ and ρ are determined by the images of the singletons,
- $X \vdash \varphi \quad \Leftrightarrow \quad \tau X \vdash' \tau\{\varphi\}$,
- $X' \vdash' \varphi' \quad \Leftrightarrow \quad \rho X' \vdash \rho\{\varphi'\}$,
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Lattice equivalence

Suppose $\vdash$ on S and $\vdash'$ on S'. They are lattice equivalent if their lattices of theories are isomorphic:

$$\mathbf{Th}(\vdash) \cong \mathbf{Th}(\vdash').$$

Questions

Question: Are these two equivalences the same?

$$\vdash \underset{\rho}{\overset{\tau}{\rightleftarrows}} \vdash' \qquad \text{if and only if} \qquad \mathbf{Th}(\vdash) \cong \mathbf{Th}(\vdash')$$

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Structurality

Let M be a monoid and $\vdash$ and $\vdash'$ substitution-invariant ADR's on M-sets S and S'. A structural translation from S to S' is a map $\tau\colon \mathcal{P}S \to \mathcal{P}S'$ so that:

- τ is determined by the images of the singletons: $\tau\Sigma = \bigcup_{\psi\in\Sigma} \tau\{\psi\}$,
- τ commutes with substitutions: $\tau\{\sigma \cdot \varphi\} = \sigma \cdot \tau\{\varphi\}$.

Galatos and Tsinakis' lifting

$M = \langle M, \star, 1\rangle$ monoid $\rightsquigarrow$ $\mathcal{A}_M = \langle \mathcal{P}M, \circ, \{1\}\rangle$

$$a \circ b = \{\sigma \star \tau : \sigma \in a,\ \tau \in b\}$$

$\cdot : M \times S \to S$ action $\rightsquigarrow$ $* : \mathcal{P}M \times \mathcal{P}S \to \mathcal{P}S$

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Module over a quantale

Fix a quantale $\mathcal{A} = \langle \mathbf{A}, \circ, 1\rangle$, which is a tuple where $\mathbf{A}$ is a complete lattice and $\circ : \mathbf{A} \times \mathbf{A} \to \mathbf{A}$ is a biresiduated map rendering $\langle A, \circ, 1\rangle$ a monoid.

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(i) $1 * x = x$,

(ii) $a * (b * x) = (a \circ b) * x$.

Closure operators on $\mathcal{A}$-modules

A closure operator on an $\mathcal{A}$-module $\mathbb{R}$ is a map $\gamma\colon R \to R$ such that

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Canonical projection of a closure operator

If γ is a closure operator on $\mathbb{R}$, define $\dot{\gamma} : R \twoheadrightarrow R_\gamma$ as $\dot{\gamma} x = \gamma x$.
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Slogan: A logic is an epimorphism in a category of modules (over a quantale).

Examples:

If $\mathcal{A}_{\mathcal{L}}$ is the quantale determined by the monoid $\mathrm{End}(\mathbf{Fm}_{\mathcal{L}})$ and $\mathbb{R}$ and $\mathbb{S}$ are the $\mathcal{A}_{\mathcal{L}}$-modules determined by $\mathrm{Fm}_{\mathcal{L}}$ and $\mathrm{Eq}_{\mathcal{L}}$, then

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- every $\mathcal{A}_{\mathcal{L}}$-morphism $\rho : \mathbb{S} \to \mathbb{R}$ is determined by a set of formulas Γ in two variables x, y in the following way: $\rho\{\delta \approx \varepsilon\} = \Gamma(\delta/x, \varepsilon/y)$.

Interpretability and representability

Def: Given $\mathcal{A}$-modules $\mathbb{R}$ and $\mathbb{S}$, and closure operators γ and δ on $\mathbb{R}$ and $\mathbb{S}$:

- an interpretation of γ into δ is a morphism $\tau : \mathbb{R} \to \mathbb{S}$, satisfying

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The Isom. Theorem hods for an $\mathcal{A}$-module $\mathbb{R}$ if and only if $\mathbb{R}$ is projective.

π-Institutions

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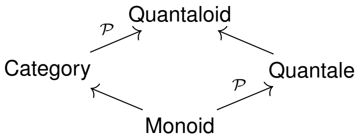
- for all $\sigma \colon \Sigma \to \Sigma'$ and $X \subseteq \mathrm{Sen}\,\Sigma$,

$$\mathrm{Sen}(\sigma)[\gamma_\Sigma(X)] \subseteq \gamma_{\Sigma'}(\mathrm{Sen}(\sigma)[X]).$$

Abstractions



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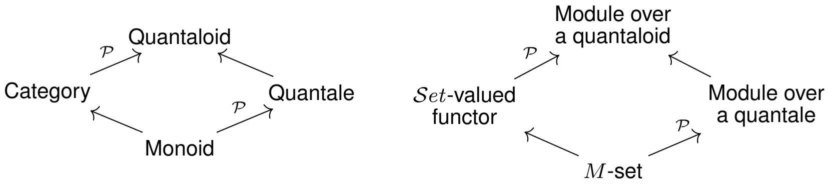

Modules over quantaloids

Def: A *quantaloid* is an enriched category $\mathcal{Q}$ over the cat. $\mathcal{S}\ell$ of sup-lattices:

- every hom-set $\mathcal{Q}(A, B)$ is a $\bigvee$-complete lattice;
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Def: A $\mathcal{Q}$-morphism between $\mathcal{Q}$-modules T and T' is a natural transformation $\tau \colon T \xrightarrow{\cdot} T'$ so that every component is residuated $\tau_A \colon TA \to T'A$.

Closure operator and its module of theories

Def: A closure operator on a $\mathcal{Q}$-module is a family $\gamma = \{\gamma_A : TA \to TA\}_{A \in \mathcal{Q}}$

- γ_A is a closure operator on the lattice TA;
- $\forall a : A \to B$ in $\mathcal{Q}, \forall x \in TA, \qquad a *_T \gamma_A x \leqslant \gamma_B(a *_T x)$. (Str)

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Given a closure operator γ on a $\mathcal{Q}$-module T, we define $T_\gamma : \mathcal{Q} \to \mathcal{S}\ell$

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Closure operator and its module of theories

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Every closure operator γ on a $\mathcal{Q}$-module T induces an epi in $\mathcal{Q}$-Mod:

$$\dot{\gamma} : T \twoheadrightarrow T_\gamma .$$

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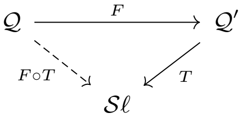

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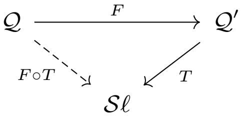

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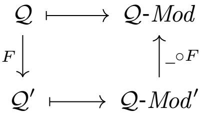

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$$\begin{array}{ccc} \pi\text{-}\mathcal{I}nst & \longrightarrow & \int \mathit{Mod} \\ {\scriptstyle \mathfrak{p}}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \mathfrak{q}} \\ \mathcal{C}at & \xrightarrow{\ \widehat{\mathcal{P}}\ } & \mathcal{S}\ell\text{-}\mathcal{C}at \end{array}$$

Thank you for your attention!